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Magnetic field due to helical currents on a torus

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Abstract. By means of coordinate transformations on different forms of the Biot–Savart law, general expressions are derived for both the magnetic vector potential and the magnetic field in two systems of toroidal coordinates. For the cases of interest, it is shown that the integrations can be performed exactly in proper toroidal coordinates, whereas approximations are necessary in quasi-toroidal coordinates. Different forms of helical windings are discussed. The results are applied to the calculation of a stellarator magnetic field of arbitrary polarity l, produced by a distribution of helical current filaments on a torus. The components of the magnetic field are given in terms of infinite series. In the intermediate region, away from the external separatrices and the central region, simple analytic expressions for the magnetic surfaces are obtained.

1. Introduction

It was suggested (Spitzer 1958) that steady state plasma confinement is possible, in principle, in magnetic field configurations produced by a distribution of helical current filaments on a torus. This was demonstrated most clearly (WVII team 1980) by the recent currentless operation of the Wendelstein VII-A stellarator, in which plasma ionisation and energy input were maintained by the injection of fast neutral atoms. In view of the practical advantages which can be accrued from a configuration which does not need a toroidal current induced in the plasma, it seems worthwhile to re-examine the confinement properties of pure toroidal traps such as torsatrons and stellarators in a totally 'non-tokamak' mode of operation. In such toroidal devices, the vacuum magnetic field plays a central role. It is the purpose of this paper to study non-axisymmetric vacuum magnetic fields inside a torus.

Early work on stellarator fields is based on linear helically symmetric systems (Johnson *et al* 1958) and toroidal effects only enter in second order in the classical stellarator expansion (Greene and Johnson 1961). In a different expansion, Dobrott and Frieman (1971) allow toroidicity to have a stronger effect on the field configuration. However, in expansion schemes, it is necessary at the outset to make explicit assumptions about the relative magnitudes of the relevant quantities such as the number of toroidal periods, the ratio of the helical field strength to the superimposed toroidal field strength etc. This limits the usefulness of such methods in parametric studies for design and optimisation purposes. Moreover, it is difficult to justify the consistency of the ordering for all regions inside the torus. For example, the expansion studies indicate that there is a single circular magnetic axis inside an $l=3$ stellarator magnetic field, whereas in fuller toroidal treatments both theoretical analysis (Aleksin 1963) and numerical computations (Gibson 1967, Blamey *et al* 1981) show that there are, in

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general, two circular magnetic axes. It seems necessary to assess toroidal effects in a more general context. A principal objective of this paper is to use toroidal coordinates to establish a general mathematical framework where toroidal effects are taken fully into account and the physical and geometrical parameters are left arbitrary.

Toroidal coordinates have been used by a number of authors to calculate magnetic fields due to helical current filaments on a torus. Kovrizhnykh (1963) took advantage of the well known solution of Laplace's equation in (proper) toroidal coordinates to obtain a magnetic scalar potential. But he averaged over the toroidal angle to obtain a two-dimensional boundary-value problem. Bhadra (1968) pointed out that the scalar potential method does not really simplify the problem, since to satisfy general boundary conditions on the toroidal surface the problem remains three dimensional. He considered the calculation of the magnetic vector potential directly from the Biot–Savart law. More systematic applications of this method to calculate the magnetic vector potential were undertaken by Tayler (1965) and Mirin *et al* (1976), who made extensive uses of expansions in toroidal harmonics. However, the calculation of the components of the magnetic field from the magnetic vector potential in toroidal coordinates still remains a rather onerous task and these authors did not present them. In this paper, the magnetic field is calculated directly from another standard form of the Biot–Savart law, which is derived in toroidal coordinates. Aleksin (1963) also used this direct approach, but he used quasi-toroidal coordinates which give rise to difficult integration problems, limiting the theory to second order in the inverse aspect ratio. The origin of this difficulty is discussed below.

In the next section, toroidal coordinates and quasi-toroidal coordinates are introduced in the standard way and coordinate transformations are discussed. These are applied to different forms of the Biot–Savart law in § 3 to derive general expressions for the magnetic vector potential and the magnetic field in both toroidal and quasi-toroidal coordinates. The relative merits of the two systems of coordinates are indicated. This is followed, in § 4, by a description of various helical windings on the torus and the relationships among them are discussed. In § 5, the formal expressions are applied specifically to the study of a stellarator field of arbitrary polarity l produced by a distribution of helical current filaments. On carrying through the integrations, the magnetic field is reduced to the evaluation of infinite series involving generalised Legendre functions. By appropriate truncation of the infinite series, approximate analytic expressions, accurate to the lowest orders of the inverse aspect ratio, are obtained. In the final section, the theoretical advances are summarised and discussed, and possible further developments and applications are indicated.

2. Toroidal coordinates

In the case of calculation of magnetic fields due to helical current filaments on the surface of a torus (with circular cross section), the mathematical simplifications that result from the use of toroidal coordinates are: (a) the description of the helical windings is simplified; (b) the labour involved in the integrations is reduced.

2.1. Coordinate systems

Two systems of orthogonal coordinates have been used to describe toroidal geometry: (a) quasi-toroidal coordinates (e.g. Mercier 1962), and (b) toroidal coordinates (e.g.

Moon and Spencer 1971). A system of quasi-toroidal coordinates (ρ, φ, ζ) can be defined (see figure 1) where the $\rho =$ constant surfaces form a set of nested concentric toroidal surfaces with respect to the circular axis $R = R_0$. These coordinates are suitable only for describing regions interior to $\rho = R_0$, since there are ambiguities when $\rho > R_0$. But because of their simple geometric interpretations in the restricted region of interest, they have been commonly used for the study of equilibrium and stability of confined toroidal plasmas. Another system of toroidal coordinates (η, θ, φ) can be defined where the $\eta =$ constant surfaces form a set of nested, but non-concentric, toroidal surfaces around the limit circle defined by $R = a$. These coordinates are applicable to the whole Euclidean space, but they are much less familiar, with a less transparent geometric interpretation.

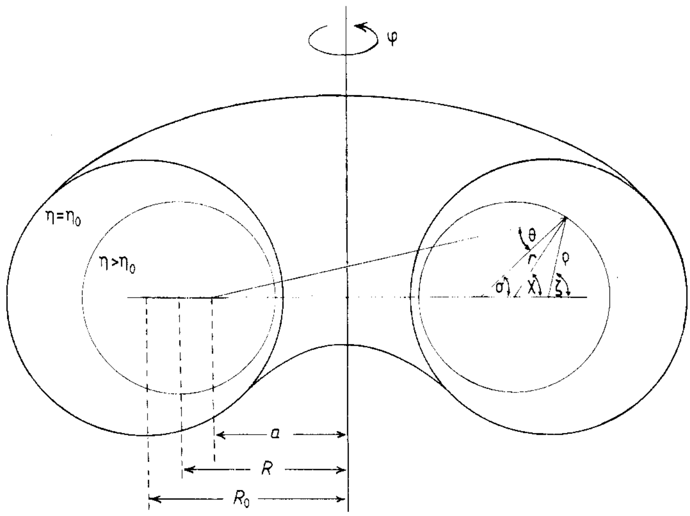

Figure 1. Cross section of toroidal surfaces showing schematic definitions of toroidal coordinates (η, θ, φ) and quasi-toroidal coordinates (ρ, φ, ζ).

The relationship between these coordinates and Cartesian coordinates can be written as follows:

$$\begin{aligned} x &= (R_0 + \rho \cos \zeta) \cos \varphi = hu \cos \varphi, \\ y &= (R_0 + \rho \cos \zeta) \sin \varphi = hu \sin \varphi, \\ z &= \rho \sin \zeta = h \sin \theta, \end{aligned} \tag{1}$$

where $h \equiv ag$, $g \equiv (v - \cos \theta)^{-1}$, $v \equiv \cosh \eta$ and $u \equiv \sinh \eta$. The differential line elements in the two systems of coordinates are

$$\mathrm{d}\boldsymbol{l} = \boldsymbol{e}_\rho \, \mathrm{d}\rho + \boldsymbol{e}_\varphi (R_0 + \rho \cos \zeta) \, \mathrm{d}\varphi + \boldsymbol{e}_\zeta \rho \, \mathrm{d}\zeta, \tag{2}$$

$$\mathrm{d}\boldsymbol{l} = h(\boldsymbol{e}_\eta \, \mathrm{d}\eta + \boldsymbol{e}_\theta \, \mathrm{d}\theta + \boldsymbol{e}_\varphi u \, \mathrm{d}\varphi), \tag{3}$$

which yield, by orthogonality, the respective metrics

$$\mathrm{d}l^2 = \mathrm{d}\rho^2 + (R_0 + \rho \cos \zeta)^2 \, \mathrm{d}\varphi^2 + \rho^2 \, \mathrm{d}\zeta^2, \tag{4}$$

$$\mathrm{d}l^2 = h^2(\mathrm{d}\eta^2 + \mathrm{d}\theta^2 + u^2 \, \mathrm{d}\varphi^2). \tag{5}$$

It is convenient for mathematical purposes to introduce auxiliary coordinates (r, χ) which are directly related to (η, θ) coordinates by the definitions

$$\begin{aligned} R/r &= \cosh\eta, & a/r &= \sinh\eta, \\ \cos\chi &= g(v\cos\theta - 1), & \sin\chi &= gu\sin\theta, \end{aligned} \tag{6}$$

where r and χ have simple and obvious geometrical interpretations (see figure 1). It follows immediately from these expressions that

$$gu^2 - \cos\chi = v, \qquad g\sin^2\theta + \cos\chi = \cos\theta. \tag{7}$$

The distance d between two points (ρ, φ, ζ) and $(\rho', \varphi', \zeta')$ is given by (Aleksin 1963)

$$d^2 = \rho^2 + \rho'^2 - 2\rho\rho'\cos(\zeta - \zeta') + 4\sin^2(\psi/2)(R_0 + \rho\cos\zeta)(R_0 + \rho'\cos\zeta'), \tag{8}$$

where $\psi = \varphi' - \varphi$. Similarly, the distance d between two points (η, θ, φ) and $(\eta', \theta', \varphi')$ is given by (Hobson 1931, Mirin *et al* 1976)

$$d^2 = 2hh'[vv' - uu'\cos\psi - \cos(\theta - \theta')], \tag{9}$$

where $h' \equiv a/(v' - \cos\theta')$, $v' \equiv \cosh\eta'$ etc.

2.2. *Coordinate transformations*

The transformation (1) between Cartesian coordinates x_i with Latin suffixes and either system of curvilinear coordinates τ_α with Greek suffixes can be expressed as

$$x_i = x_i(\tau_\alpha). \tag{10}$$

On defining h_α $(\alpha = 1, 2, 3)$ by $dl^2 \equiv h_1^2\, d\tau_1^2 + h_2^2\, d\tau_2^2 + h_3^2\, d\tau_3^2$, standard transformation theory can be used to show that the matrix $J_{i\alpha}$ defined by

$$J_{i\alpha} \equiv (1/h_\alpha)(\partial x_i/\partial\tau_\alpha) \qquad \text{(no summation)} \tag{11}$$

is orthogonal:

$$J_{\alpha i}J_{i\beta} = \delta_{\alpha\beta}, \qquad J_{i\alpha}J_{\alpha j} = \delta_{ij}, \tag{12}$$

where $\delta_{\alpha\beta}$ and δ_{ij} are the Kronecker deltas and the summation convention is implied unless stated otherwise. The orthonormal basis vectors $\boldsymbol{e}_i$ in Cartesian coordinates are related to the orthonormal basis vectors $\boldsymbol{e}_\alpha$ used in (2) and (3) by

$$\boldsymbol{e}_i = J_{i\alpha}\boldsymbol{e}_\alpha, \qquad \boldsymbol{e}_\alpha = J_{\alpha i}\boldsymbol{e}_i. \tag{13}$$

Hence for any arbitrary vector $\boldsymbol{A} = A_\alpha\boldsymbol{e}_\alpha = A_i\boldsymbol{e}_i$, (13) gives the transformation laws for the components,

$$A_\alpha = J_{i\alpha}A_i, \qquad A_i = J_{\alpha i}A_\alpha. \tag{14}$$

Direct evaluation of (11), using definitions (1) and (10), shows that for quasi-toroidal coordinates

$$J_{i\alpha} = \begin{pmatrix} \cos\zeta\cos\varphi & -\sin\varphi & -\sin\zeta\cos\varphi \\ \cos\zeta\sin\varphi & \cos\varphi & -\sin\zeta\sin\varphi \\ \sin\zeta & 0 & \cos\zeta \end{pmatrix}, \tag{15}$$

whilst for toroidal coordinates

$$J_{i\alpha} = \begin{pmatrix} -\cos\chi\cos\varphi & -\sin\chi\cos\varphi & -\sin\varphi \\ -\cos\chi\sin\varphi & -\sin\chi\sin\varphi & \cos\varphi \\ -\sin\chi & \cos\chi & 0 \end{pmatrix}. \tag{16}$$

The elements of (16) were also written down by Mirin *et al* (1976), but without the convenient benefit of the angle χ. Here, the orthogonality of $J_{i\alpha}$ is easily verified.

3. Biot–Savart law

The magnetic vector potential $\boldsymbol{A}$ defined by $\boldsymbol{B} = \nabla\times\boldsymbol{A}$ and $\nabla\cdot\boldsymbol{A} = 0$ satisfies the differential equation

$$\nabla^2\boldsymbol{A} = -\mu_0\boldsymbol{j}, \tag{17}$$

where $\boldsymbol{j}$ is the source current density and SI units are used. Only in Cartesian coordinates is this equation separable, and it has a formal solution

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int\frac{\boldsymbol{j}(\boldsymbol{r}')\,\mathrm{d}^3\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|}, \tag{18}$$

where $|\boldsymbol{r}-\boldsymbol{r}'|^{-1}$ is the Green function and the integration is performed over all points $\boldsymbol{r}'$ of the source current density. If the source current density has a singular distribution in the form of a current filament of total current I, then (18) becomes

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0 I}{4\pi}\int\frac{\mathrm{d}\boldsymbol{l}'}{|\boldsymbol{r}-\boldsymbol{r}'|}, \tag{19}$$

where the line integration is now taken along the current filament. From this, the magnetic field (more correctly the magnetic induction) is given by

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0 I}{4\pi}\int\frac{\mathrm{d}\boldsymbol{l}'\times(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}. \tag{20}$$

Both (19) and (20) have been called the Biot–Savart law. It is clear from the above derivation that they are valid only in Cartesian coordinates. On account of the toroidal geometry of the current filaments, it is convenient to re-express the Biot–Savart law in quasi-toroidal and toroidal coordinates.

3.1. Magnetic vector potential

Application of the transformation laws (14)–(19) show

$$A_\alpha = \frac{\mu_0 I}{4\pi}\int\frac{U_{\alpha\beta}\,\mathrm{d}l'_\beta}{d}, \tag{21}$$

where $d = |\boldsymbol{r}-\boldsymbol{r}'|$ and the tensor $U_{\alpha\beta}$ is defined by

$$U_{\alpha\beta} \equiv J_{\alpha i}J'_{i\beta} = J_{\alpha i}(\boldsymbol{r})J_{i\beta}(\boldsymbol{r}'). \tag{22}$$

Calculations from (15) show that, in the case of quasi-toroidal coordinates,

$$U_{\alpha\beta}=\begin{pmatrix} \cos\zeta\cos\zeta'\cos\psi+\sin\zeta\sin\zeta' & -\cos\zeta\sin\psi & -\cos\zeta\sin\zeta'\cos\psi+\sin\zeta\cos\zeta' \\ \cos\zeta'\sin\psi & \cos\psi & -\sin\zeta'\sin\psi \\ -\sin\zeta\cos\zeta'\cos\psi+\cos\zeta\sin\zeta' & \sin\zeta\sin\psi & \sin\zeta\sin\zeta'\cos\psi+\cos\zeta\cos\zeta' \end{pmatrix}, \tag{23}$$

where $\psi=\varphi'-\varphi$. Similarly, application of (16) shows that in the case of toroidal coordinates

$$U_{\alpha\beta}=\begin{pmatrix} \cos\chi\cos\chi'\cos\psi+\sin\chi\sin\chi' & \cos\chi\sin\chi'\cos\psi-\sin\chi\cos\chi' & \cos\chi\sin\psi \\ \sin\chi\cos\chi'\cos\psi-\cos\chi\sin\chi' & \sin\chi\sin\chi'\cos\psi+\cos\chi\cos\chi' & \sin\chi\sin\psi \\ -\cos\chi'\sin\psi & -\sin\chi'\sin\psi & \cos\psi \end{pmatrix}. \tag{24}$$

It is apparent from (2) and (3) that, if the current filament lies on a toroidal surface, then $dl'_\rho=0$ and $dl'_\eta=0$ and the number of integrals in (21) is reduced.

If the current filament lying on the $\rho=\rho_0$ surface winds according to the relationship $d\zeta'/d\varphi'=\nu(\varphi')$ then equations (2), (21) and (23) can be used to show

$$A_\alpha=\frac{\mu_0 I}{4\pi}\int\frac{a_\alpha\, d\varphi'}{d}, \tag{25}$$

where d is determined by (8) and

$$\begin{aligned} a_\rho&=-\bar{R}'\cos\zeta\sin\psi+\rho_0\nu(\sin\zeta\cos\zeta'-\cos\zeta\sin\zeta'\cos\psi),\\ a_\varphi&=\bar{R}'\cos\psi-\rho_0\nu\sin\zeta'\sin\psi,\\ a_\zeta&=\bar{R}'\sin\zeta\sin\psi+\rho_0\nu(\cos\zeta\cos\zeta'+\sin\zeta\sin\zeta'\cos\psi), \end{aligned} \tag{26}$$

with $\bar{R}'\equiv R_0+\rho_0\cos\zeta'$ and $\zeta'=\int^{\varphi'}\nu(\bar\varphi)\,d\bar\varphi$. In the same way, if the winding law of the current filament on the $\eta=\eta_0$ surface is determined by $d\theta'/d\varphi'=\nu(\varphi')$, where $\nu(\varphi')$ is again arbitrary, then the components of the magnetic vector potential in toroidal coordinates are given also formally by (25), but d is now given by (9) and

$$\begin{aligned} a_\eta&=\nu(\cos\chi\sin\chi'\cos\psi-\sin\chi\cos\chi')+u'\cos\chi\sin\psi,\\ a_\theta&=\nu(\sin\chi\sin\chi'\cos\psi+\cos\chi\cos\chi')+u'\sin\chi\sin\psi,\\ a_\varphi&=-\nu\sin\chi'\sin\psi+u'\cos\psi, \end{aligned} \tag{27}$$

where $\psi=\varphi'-\varphi$ as previously. Only the A_φ component in toroidal coordinates was obtained by Bhadra (1968).

3.2. Magnetic field

Although the magnetic vector potential $\boldsymbol{A}$, being related to certain invariants of the magnetic field configuration, is of considerable interest in itself, to calculate the magnetic field $\boldsymbol{B}$ from $\boldsymbol{A}$ requires a further vector differentiation in $\boldsymbol{B}=\nabla\times\boldsymbol{A}$. In principle, this is straightforward, but in practice the expression for $\boldsymbol{A}$ is already so complicated on integration that the procedure is prohibitively cumbersome. Apparently no author has carried this calculation to any detail. An alternative method pursued here is to use the Biot–Savart law in the form given by (20), which is tantamount to performing the vector differentiation before carrying out the integration over the source currents.

Again, application of transformation laws (14)–(20) shows

$$B_\alpha = \frac{\mu_0 I}{4\pi} \int \frac{W_{\alpha\beta}\, dl'_\beta}{d^3}, \tag{28}$$

where $d = |\boldsymbol{r} - \boldsymbol{r}'|$ and

$$W_{\alpha\beta} \equiv \varepsilon_{ijk} J_{i\alpha} J'_{\beta j} (x_k - x'_k), \tag{29}$$

with ε_{ijk} being the permutation symbol. After some algebraic manipulations it can be shown that

$$W_{\alpha\beta} = W_{\alpha\beta}(\boldsymbol{r}, \boldsymbol{r}') = T_{\alpha\beta}(\boldsymbol{r}, \boldsymbol{r}') + T_{\beta\alpha}(\boldsymbol{r}, \boldsymbol{r}'). \tag{30}$$

In the case of quasi-toroidal coordinates, on defining

$$\bar{R} = R_0 + \rho \cos \zeta, \qquad \bar{R}' = R_0 + \rho' \cos \zeta', \tag{31}$$

calculations show

$$T_{\alpha\beta}(\boldsymbol{r}, \boldsymbol{r}') = \begin{pmatrix} -R_0 \sin \zeta \cos \zeta' \sin \psi & -\rho \sin \zeta \cos \zeta' \cos \psi + \bar{R} \sin \zeta' & -(R_0 \cos \zeta + \rho) \cos \zeta' \sin \psi \\ -R_0 \sin \zeta \cos \psi & \rho \sin \zeta \sin \psi & -(R_0 \cos \zeta + \rho) \cos \psi \\ R_0 \sin \zeta \sin \zeta' \sin \psi & \rho \sin \zeta \sin \zeta' \cos \psi + \bar{R} \cos \zeta' & (R_0 \cos \zeta + \rho) \sin \zeta' \sin \psi \end{pmatrix}, \tag{32}$$

where $\psi = \varphi' - \varphi$. Similarly, in the case of toroidal coordinates,

$$T_{\alpha\beta}(\boldsymbol{r}, \boldsymbol{r}') = h \times \begin{pmatrix} -v \sin \theta \cos \chi' \sin \psi & -v \sin \theta \sin \chi' \sin \psi & v \sin \theta \cos \psi \\ u \cos \theta \cos \chi' \sin \psi & u \cos \theta \sin \chi' \sin \psi & -u \cos \theta \cos \psi \\ \sin \theta \cos \chi' \cos \psi - u \sin \chi' & \sin \theta \sin \chi' \cos \psi + u \cos \chi' & \sin \theta \sin \psi \end{pmatrix}, \tag{33}$$

where uses have been made of relationships (7). Hence, on using equations (2) and (28)–(32), the magnetic field which corresponds to the vector potential given by (25) and (26) reads

$$B_\alpha = \frac{\mu_0 I}{4\pi} \int \frac{b_\alpha\, d\varphi'}{d^3}, \tag{34}$$

where

$$\begin{aligned} b_\rho &= \bar{R}'[\bar{R}' \sin \zeta - (R_0 \sin \zeta + \rho_0 \cos \zeta \sin \zeta') \cos \psi] \\ &\qquad + \rho_0 \nu \sin \psi [R_0 \cos(\zeta - \zeta') + \rho_0 \cos \zeta], \\ b_\zeta &= \bar{R}'[\bar{R}' \cos \zeta - (R_0 \cos \zeta - \rho_0 \sin \zeta \sin \zeta' + \rho) \cos \psi] \\ &\qquad - \rho_0 \nu \sin \psi [R_0 \sin(\zeta - \zeta') + \rho_0 \sin \zeta - \rho \sin \zeta'], \\ b_\varphi &= \bar{R}' \sin \psi (\rho \sin \zeta - \rho_0 \sin \zeta') - \rho_0 \nu [\bar{R} \cos \zeta' + (\rho \sin \zeta \sin \zeta' - R_0 \cos \zeta' - \rho_0) \cos \psi], \end{aligned} \tag{35}$$

with $\bar{R} = R_0 + \rho \cos \zeta$, $\bar{R}' = R_0 + \rho_0 \cos \zeta'$ and d determined by (8). Equivalent expressions were written down by Aleksin (1963), whose paper contains, however, a number of serious misprints, particularly in the expression for the B_φ component. Similarly, on using (3), (28)–(30) and (33), the magnetic field which corresponds to the vector potential given by (25) and (27) can be shown, after some algebra, to read

$$B_\alpha = \frac{\mu_0 I}{8\pi a \sqrt{2}} u'(v - \cos \theta)^{1/2} \int \frac{b_\alpha\, d\varphi'}{S}, \tag{36}$$

where $S=(v'-\cos\theta')^{1/2}[vv'-uu'\cos\psi-\cos(\theta-\theta')]^{3/2}$,

$$b_\eta=\sin\theta(vv'\cos\psi-uu')-v[\nu\cos(\theta-\theta')\sin\psi+\sin(\theta-\theta')\cos\psi]$$
$$+\nu\cos\theta'\sin\psi-\sin\theta'\cos\psi,$$
$$b_\theta=\cos\theta(u'v-uv'\cos\psi)-u[\nu\sin(\theta-\theta')\sin\psi-\cos(\theta-\theta')\cos\psi]-u', \tag{37}$$
$$b_\varphi=\cos\theta'(\nu/u')(v'u-vu'\cos\psi)+(v'\sin\theta-v\sin\theta')\sin\psi$$
$$+\nu\cos(\theta-\theta')\cos\psi-\sin(\theta-\theta')\sin\psi-\nu u/u',$$

with $\psi=\varphi'-\varphi$. This result, which does not seem to have been derived in the literature before, will be shown to lead to a tractable method for calculating stellarator-type magnetic fields. The principal difficulty in the calculation of the magnetic field due to a helical current filament on a torus resides in the remaining integration, which is made difficult by the nature of the denominators of the integrand. The only known method is apparently to expand the integrand in terms of infinite series of generalised Legendre functions, with the angular variables appearing as harmonics. A fundamental expansion theorem (Hobson 1931, Robin 1958) reads

$$(\mu-\cos\psi)^{-1/2}=\frac{2\sqrt{2}}{\pi}\sum_{n=0}^{\infty}\varepsilon_n Q_{n-1/2}(\mu)\cos n\psi, \tag{38}$$

where the Neumann symbol is defined by $\varepsilon_0=\frac{1}{2}$ and $\varepsilon_n=1$ otherwise. In the case of quasi-toroidal coordinates, to use (38), (8) is written in the form

$$d^{-1}=D^{-1}(\mu-\cos\psi)^{-1/2}, \tag{39}$$

where $D^2\equiv 2(R_0+\rho\cos\zeta)(R_0+\rho'\cos\zeta')$ and $\mu=1+\Delta/D^2$, with $\Delta\equiv\rho^2+\rho'^2-2\rho\rho'\cos(\zeta-\zeta')$. It is evident that the argument of the Legendre function $Q_{n-1/2}(\mu)$ appearing in (38) is quite complicated in this case. In order to perform the integration, the Legendre function $Q_{n-1/2}(\mu)$ needs to be further expanded to make the angular dependence explicit. It was found necessary (Aleksin 1963) to make the approximation $D^2\simeq 2R_0^2$ so that the integration is tractable. This limits the accuracy of the theory to second order in the inverse aspect ratio.

On the other hand, in the case of toroidal coordinates, due to the applicability of an addition theorem, the denominator can be expanded exactly. The use of (38) shows

$$[vv'-uu'\cos\psi-\cos(\theta-\theta')]^{-1/2}=\frac{2\sqrt{2}}{\pi}\sum_{n=0}^{\infty}\varepsilon_n Q_{n-1/2}(\mu)\cos n(\theta-\theta'), \tag{40}$$

where in this case $\mu\equiv vv'-uu'\cos\psi$. The relevant addition theorem (Hobson 1931) reads

$$Q_{n-1/2}(\mu)=\sum_{s=0}^{\infty}\varepsilon_s(-1)^s Q^s_{n-1/2}(v)P^{-s}_{n-1/2}(v')\cos s\psi \qquad (v>v'), \tag{41}$$

where $P^{-s}_{n-1/2}$ and $Q^s_{n-1/2}$ are generalised associated Legendre functions of the first and second kind respectively. These expansions were used by Tayler (1965) and Mirin *et al* (1976) in their calculation of the magnetic vector potential. To use these to calculate the magnetic field, however, one needs to proceed a little further, on account of the form of S in (36). The differentiation of (40) with respect to μ yields

$$[vv'-uu'\cos\psi-\cos(\theta-\theta')]^{-3/2}=-\frac{4\sqrt{2}}{\pi}\sum_{n=0}^{\infty}\varepsilon_n\dot{Q}_{n-1/2}(\mu)\cos n(\theta-\theta'), \tag{42}$$

where the dot denotes derivative with respect to the argument. Moreover, it is easy to see

$$\dot{Q}_{n-1/2}(\mu)=\frac{1}{v'}\frac{\lambda}{\lambda-\cos\psi}\frac{\partial}{\partial v}Q_{n-1/2}(\mu), \tag{43}$$

where $\lambda \equiv v'u/vu'$. It follows from the definitions $v \equiv \cosh\eta$ etc that $\lambda > 1$, whenever $\eta > \eta'$. Applying the fundamental expansion theorem (38) to the factor $(v' - \cos\theta')^{-1/2}$ appearing in S^{-1} in (36) and collecting the results (41)–(43), one finds an exact expansion for the denominator of (36):

$$S^{-1}=\sum_{m,n,s=0}^{\infty} a^{s}_{mn}\cos m\theta'\cos n(\theta'-\theta)\frac{\cos s\psi}{\lambda-\cos\psi}, \tag{44}$$

where the coefficients a^{s}_{mn} are defined by

$$a^{s}_{mn}\equiv-(16/\pi^{2})(\lambda/v')(-1)^{s}\varepsilon_m\varepsilon_n\varepsilon_s Q_{m-1/2}(v')\dot{Q}^{s}_{n-1/2}(v)P^{-s}_{n-1/2}(v'). \tag{45}$$

Hence the problem of calculating the magnetic field due to a helical current filament given by (36) has been reduced to elementary integrals over trigonometric functions. Since the expansions are exact, there is no limit to the accuracy of the method.

4. Helical windings

A straight circular cylinder has a natural central axis and hence a uniform helical winding of arbitrary constant pitch has a clear and unambiguous meaning. Moreover, such windings are members of the family of geodesic curves on the cylindrical surface. In the case of a torus, depending on how the poloidal angle is defined, there is more than one definition for a uniform helical winding. Furthermore, on account of another periodicity in the toroidal direction, a single helix of arbitrary constant pitch can cover the toroidal surface without closing on itself. For practical reasons, a helical current winding is required to close upon itself after a finite number of turns around the torus. There are then geometric limitations on toroidal helical current windings.

4.1. Geodesics

From a certain mathematical point of view, geodesics on the torus form a natural analogue to uniform helices on a straight circular cylinder (Tayler 1965). Their equations may be found directly by taking variations of the line element given by either (4) or (5). In toroidal coordinates, (5) can be written

$$\mathrm{d}l=h[1+u^{2}(\mathrm{d}\varphi/\mathrm{d}\theta)^{2}]^{1/2}\,\mathrm{d}\theta\equiv F(\theta,\mathrm{d}\varphi/\mathrm{d}\theta)\,\mathrm{d}\theta. \tag{46}$$

From the variational principle $\delta\int \mathrm{d}l=0$, the Euler–Lagrange equation, which here simply reads $\partial F/\partial(\mathrm{d}\varphi/\mathrm{d}\theta)=$ constant, yields the geodesic equation

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\theta}=\frac{K/u}{(h^{2}u^{2}-K^{2})^{1/2}}, \tag{47}$$

where K is a constant. Equivalently, in quasi-toroidal coordinates, the geodesic equation reads

$$\frac{d\varphi}{d\zeta}=\frac{K\rho_0}{(R_0+\rho_0\cos\zeta)[(R_0+\rho_0\cos\zeta)^2-K^2]^{1/2}},\tag{48}$$

where K is a constant with $|K|<R_0-\rho_0$. On substitution $t=K/(R_0+\rho_0\cos\zeta)$, it can be shown that

$$\frac{d\varphi}{dt}=\frac{\rho_0 t}{a[(1-t)(b-t)(t-c)(t+1)]^{1/2}},\tag{49}$$

where $b\equiv K/(R_0-\rho_0)$ and $c\equiv K/(R_0+\rho_0)$. The solution of this equation can be expressed as the sum of two incomplete elliptic integrals (Gradshteyn and Ryzhik 1965, p 243). Hence, in general, the geodesic curves on the torus have a rather complicated mathematical form. On introducing a constant ν by

$$K\rho_0\nu\equiv R_0(R_0^2-K^2)^{1/2},\tag{50}$$

it can be shown that provided $\nu^2\gg 2R_0/\rho_0\gg 1$, the right-hand side of (48) can be expressed as a rapidly converging series,

$$\frac{d\varphi}{d\zeta}=\frac{1}{\nu}\left[1-\left(\frac{2\rho_0}{R_0}+\frac{R_0}{\rho_0\nu^2}\right)\cos\zeta+\ldots\right],\tag{51}$$

which on integration yields

$$\nu(\varphi-\varphi_0)=\zeta-\left(\frac{2\rho_0}{R_0}+\frac{R_0}{\rho_0\nu^2}\right)\sin\zeta+\ldots,\tag{52}$$

where φ_0 is a phase constant. It is evident that these curves approach the geodesics of a straight circular cylinder, provided the pitch length is sufficiently short, $\nu^2\gg 2R_0/\rho_0$, to overcome the effects of toroidal curvature.

4.2. *Uniform helices*

In quasi-toroidal coordinates, the poloidal angle ζ is defined with respect to the central axis of the toroidal chamber. Hence a natural definition for a uniform helix in this system of coordinates reads

$$d\zeta/d\varphi=\nu=\text{constant}.\tag{53}$$

By comparison with (51), it can be seen that this equation may be regarded as the geodesic equation in the limit of zero toroidal curvature. Indeed, the equation for a helix on a straight circular cylinder of period length L may be cast in the form (53), provided one identifies $L=2\pi R_0/\nu$. The classical helical windings, and in particular the classical stellarator windings, obey this relationship (Aleksin 1963, Mirin *et al* 1976). It is clear from the above discussion that such windings are really cylindrical windings imposed on the torus, with the additional condition that ν must be a rational number for closure of the helical windings.

In toroidal coordinates, on the other hand, the natural definition for a uniform helix reads

$$d\theta/d\varphi=\nu=\text{constant}.\tag{54}$$

Such windings were considered by Tayler (1965), and they are the windings of the 'force-free' toroidal solenoid (Koryavko and Litvinenko 1979). Unlike those defined by (48) and (53), such helices incline at constant angles to the generatrix of the toroidal surface. On account of this and other mathematical properties of the windings, and in order to distinguish them from classical windings, these will be called canonical windings. For example, it may be shown from (1) that for the special case $\nu = 1$, a canonical helical winding on the torus can be described by the equation

$$(x-\rho_0)^2+y^2+z^2=R_0^2. \tag{55}$$

That is, the curve is a circle of radius R_0, centred at $x=\rho_0$, $y=z=0$, with its plane tilted to the equatorial plane at an angle $\sin^{-1}(\rho_0/R_0)$. Such circles are called Villarceau circles by Gourdon *et al* (1968), who studied numerically the magnetic fields of stellarator and torsatron configurations formed by such circles.

The above two classes of uniform helices by no means exhaust all the reasonable possibilities. A poloidal angle σ (see figure 1) may be defined with respect to the limiting circle of the system of toroidal coordinates. From geometric relationships, it may be shown (appendix 1) that on the $\rho=\rho_0$ toroidal surface

$$2\sigma=\theta+\zeta. \tag{56}$$

Hence a uniform helix defined by $\mathrm{d}\sigma/\mathrm{d}\varphi=$ constant can be regarded as an arithmetic mean between those defined by (53) and (54).

Evidently, classical windings (53) may be expressed in toroidal coordinates and conversely canonical windings (54) may be expressed in quasi-toroidal coordinates. From the definitions (6), it can be shown (appendix 1) that on the toroidal surface $\rho=\rho_0$ or $\eta=\eta_0$

$$\theta=2\tan^{-1}\left(\frac{R_0-\rho_0}{a}\tan\frac{\zeta}{2}\right)=\zeta+2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\alpha^n\sin n\zeta, \tag{57}$$

where $\alpha\equiv[(R_0-a)/(R_0+a)]^{1/2}=[1-(1-\varepsilon^2)^{1/2}]/\varepsilon$, with $\varepsilon=\rho_0/R_0$ being the inverse aspect ratio of the toroidal chamber. Hence, in quasi-toroidal coordinates, the canonical winding relation (54) may be written as

$$\nu(\varphi-\varphi_0)=\zeta-2\alpha\sin\zeta+\alpha^2\sin 2\zeta-\ldots, \tag{58}$$

where φ_0 is a phase constant and $\alpha\ll 1$ usually. Thus a canonical helical winding appears as a specific type of modulation of the classical helical winding. Clearly the modulation can be generalised by allowing the multiplicative constants of the harmonic functions to take values not necessarily those in (58). By optimisation of certain properties of the magnetic field configuration, Gourdon *et al* (1970) have suggested certain functional forms for these multiplicative factors, which give rise to the so-called ultimate winding. In the remainder of this paper only canonical windings are considered; other windings may be considered as modulations of these.

5. A canonical stellarator field

The formulae of previous sections are now applied to the calculation of a stellarator magnetic field due to a distribution of current filaments whose winding relationships satisfy (54). A stellarator of polarity l has l pairs of helical conductors with alternating directions of currents, together with a superimposed toroidal magnetic field. In practice, in order to carry an appreciable current, the helical conductors are not filamentary but have finite physical dimensions. Partly to simulate this reality, and partly to simplify the mathematics, we consider a sinusoidal distribution of helical current filaments closely spaced on the torus. In effect, the current (I) passing through a given meridonal plane, which can be represented by a sequence of positive and negative step functions, is now replaced by a sinusoidal function in the poloidal angle (θ) (see figure 2). For such a distribution of infinitesimally thin filaments, the current passing

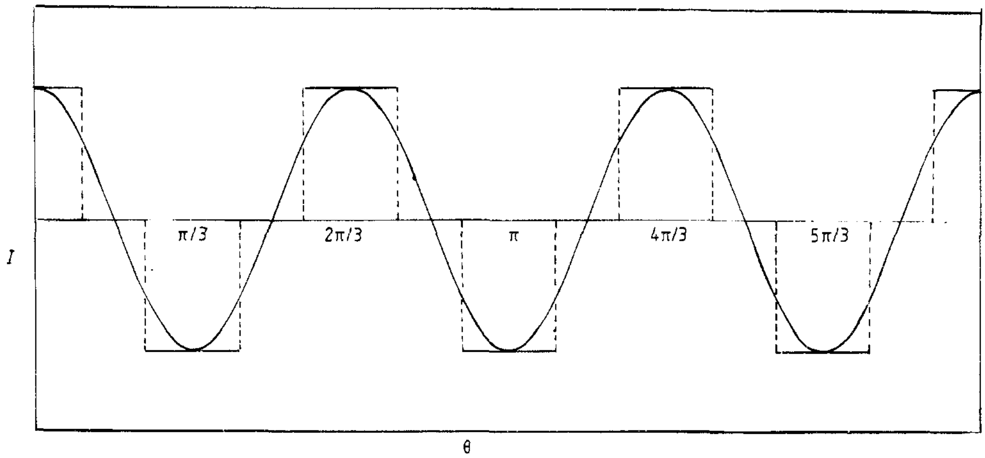

Figure 2. Current distributions on a meridonal plane. The step functions are replaced by a sinusoidal function.

through an arc $\mathrm{d}\theta_0'$ on a meridonal plane can be written as

$$\mathrm{d}I = I_0 \cos l\theta_0' \, \mathrm{d}\theta_0', \tag{59}$$

where the solution of (54) is written in the form $\theta_0' = \theta' - \nu\varphi'$ on the chamber surface and I_0 is a constant. It is evident from (59) that there is zero net current flowing through each meridonal plane, a fact which makes a modular construction of stellarator windings possible. On summing up all current filaments specified by (59), the resultant magnetic field from (28) reads

$$B_\alpha = \frac{\mu_0 I_0}{4\pi} \int_0^{2\pi} \mathrm{d}\theta_0' \cos l\theta_0' \oint \frac{W_{\alpha\beta}\, \mathrm{d}l_\beta'}{d^3}, \tag{60}$$

where the line integral goes once around the toroidal direction. It is convenient to change the variable of integration from θ_0' to θ'. The use of (36) in (60) shows that the resultant magnetic field can be expressed as

$$B_\alpha = \frac{\mu_0 I_0}{8\pi a\sqrt{2}} u'(v - \cos\theta)^{1/2} \iint_0^{2\pi} \mathrm{d}\theta' \, \mathrm{d}\varphi' \frac{\cos l(\theta' - \nu\varphi') b_\alpha}{S}, \tag{61}$$

where b_α and S are given by (37). The closure of the current windings required ν to be a rational number. If $\nu = p/l$, where p is an integer incommensurate with l, then a single filament makes p turns the short way and closes on itself after l turns the long way around the torus. In some special cases, such as the case of the Villarceau circle where $\nu = 1$, ν is an integer. In these cases, a single filament closes on itself after only one turn the long way around the torus. Irrespective of whether ν is an integer or a rational number, equation (61) is applicable generally to a canonical stellarator of arbitrary l.

5.1. A series representation

Proceeding with the integrations in (61), one conveniently rewrites it in the form

$$B_\alpha = (\mu_0 I_0/8\pi a\sqrt{2})u'(v-\cos\theta)^{1/2}(\cos\Phi B^{c}_\alpha + \sin\Phi B^{s}_\alpha), \tag{62}$$

where $\Phi = l\theta - p\varphi$ and

$$\begin{pmatrix} B^{c}_\alpha \\ B^{s}_\alpha \end{pmatrix} = \iint_0^{2\pi} \frac{d\theta'\, d\psi\, b_\alpha}{S}\begin{pmatrix} \cos l(\theta'-\theta) & \sin l(\theta'-\theta) \\ -\sin l(\theta'-\theta) & \cos l(\theta'-\theta)\end{pmatrix}\begin{pmatrix}\cos p\psi \\ \sin p\psi\end{pmatrix}. \tag{63}$$

From (37), (44) and (63), it is seen that it is simplest to do first the ψ-integrations which have the basic elementary form

$$\int_0^{2\pi} \frac{\cos N\psi}{\lambda - \cos\psi}\, d\psi = \frac{2\pi \exp(-N\beta)}{\sinh\beta}, \tag{64}$$

where one introduces $\beta \equiv -\ln[\lambda - (\lambda^2-1)^{1/2}] = \ln[\lambda + (\lambda^2-1)^{1/2}]$. It should be noted also that $\cosh\beta = \lambda = v'u/vu'$ and $\sinh\beta = (\lambda^2-1)^{1/2} = (v^2-v'^2)^{1/2}/vu'$. On performing the ψ-integrations, (63) can be re-expressed as

$$\begin{pmatrix} B^{c}_\alpha \\ B^{s}_\alpha \end{pmatrix} = \int_0^{2\pi} d\theta'\, \hat{S}^{-1}\begin{pmatrix} \cos l(\theta'-\theta) & \sin l(\theta'-\theta) \\ -\sin l(\theta'-\theta) & \cos l(\theta'-\theta)\end{pmatrix}\begin{pmatrix}\cosh(p\beta)\hat{b}^{c}_\alpha \\ \sinh(p\beta)\hat{b}^{s}_\alpha\end{pmatrix}, \tag{65}$$

where one defines

$$\hat{S}^{-1} = 2\pi \sum_{m,n,s=0}^{\infty} a^{s}_{mn} \cos m\theta' \cos n(\theta'-\theta)\exp(-s\beta), \tag{66}$$

and

$$\begin{aligned}
\hat{b}^{c}_\eta &= u(v^2-v'^2)^{-1/2}[uv'\sin(\theta'-\theta) + v\sin\theta - v'\sin\theta'],\\
\hat{b}^{c}_\theta &= (v^2-v'^2)^{-1/2}[u^2v'\cos(\theta'-\theta) - vu'^2 - \cos\theta(v^2-v'^2)],\\
\hat{b}^{c}_\varphi &= \nu u(v^2-v'^2)^{-1/2}[v'\cos(\theta'-\theta) - v],\\
\hat{b}^{s}_\eta &= \nu[v\cos(\theta'-\theta) - \cos\theta'],\\
\hat{b}^{s}_\theta &= -\nu u\sin(\theta'-\theta),\\
\hat{b}^{s}_\varphi &= v\sin\theta' - v'\sin\theta - \sin(\theta'-\theta).
\end{aligned} \tag{67}$$

The remaining θ'-integrations are elementary. On introducing, for brevity, the coefficients

$$a^{m}_{n} = \sum_{s=0}^{\infty} a^{s}_{mn}\exp(-s\beta), \tag{68}$$

and defining $\tau = \theta'-\theta$, the θ'-integrals are of the basic types

$$\begin{Bmatrix} I_1 \\ I_2 \\ I_3 \end{Bmatrix} \equiv 2\pi \sum_{m,n=0}^{\infty} a^{m}_{n} \int_0^{2\pi} \cos m(\tau+\theta)\cos n\tau \cos l\tau \begin{Bmatrix} 1 \\ \cos\tau \\ \sin\tau \end{Bmatrix} d\tau, \tag{69}$$

$$\begin{Bmatrix} J_1 \\ J_2 \\ J_3 \end{Bmatrix} = 2\pi \sum_{m,n=0}^{\infty} a_n^m \int_0^{2\pi} \cos m(\tau+\theta) \cos n\tau \sin l\tau \begin{Bmatrix} 1 \\ \cos\tau \\ \sin\tau \end{Bmatrix} \mathrm{d}\tau. \tag{70}$$

Direct elementary integration shows

$$\begin{aligned}
I_1 &= \pi^2 \sum_{m,n} a_n^m \cos m\theta \sum_{\pm} \delta_n^{\pm l \pm m}, \\
I_2 &= \frac{\pi^2}{2} \sum_{m,n} a_n^m \cos m\theta \sum_{\pm} \delta_n^{\pm l \pm m \pm 1}, \\
I_3 &= \frac{\pi^2}{2} \sum_{m,n} a_n^m \sin m\theta \sum_{\pm} (\delta_n^{\pm l \pm (m-1)} - \delta_n^{\pm l \pm (m+1)}), \\
J_1 &= \pi^2 \sum_{m,n} a_n^m \sin m\theta \sum_{\pm} (\delta_n^{\pm (l+m)} - \delta_n^{\pm (l-m)}), \\
J_2 &= \frac{\pi^2}{2} \sum_{m,n} a_n^m \sin m\theta \sum_{\pm} (\delta_n^{\pm (l-m) \pm 1} - \delta_n^{\pm (l-m) \pm 1}), \\
J_3 &= \frac{\pi^2}{2} \sum_{m,n} a_n^m \cos m\theta \sum_{\pm} (\delta_n^{\pm (l-1) \pm m} - \delta_n^{\pm (l+1) \pm m}),
\end{aligned} \tag{71}$$

where the last sum in each I_i and J_i $(i=1,2,3)$ indicates taking all combinations of signs. By definition, l, m and n are positive and hence some of the Kronecker deltas vanish necessarily. The components B_α^c and B_α^s are now seen as functions of the six integrals given by (71),

$$B_\alpha^c = B_\alpha^c(I_i, J_i), \qquad B_\alpha^s = B_\alpha^s(I_i, J_i). \tag{72}$$

From the mathematical structure of (65), it can be seen that

$$B_\alpha^s(I_i, J_i) = B_\alpha^c(-J_i, I_i). \tag{73}$$

That is, $B_\alpha^s(I_i, J_i)$ can be obtained from $B_\alpha^c(I_i, J_i)$ on replacing each I_i by $-J_i$ and each J_i by I_i $(i=1,2,3)$. It suffices therefore to record $B_\alpha^c(I_i, J_i)$, which can be found from applying the definitions (69) and (70) to (65)–(68). These read

$$\begin{aligned}
B_\eta^c &= u(v^2 - v'^2)^{-1/2} \cosh p\beta [vv' I_3 + vI_1 \sin\theta - v'(I_3 \cos\theta + I_2 \sin\theta)] \\
&\quad + \nu \sinh p\beta (vJ_2 - J_2 \cos\theta + J_3 \sin\theta), \\
B_\theta^c &= (v^2 - v'^2)^{-1/2} \cosh p\beta [u^2 v' I_2 - vu'^2 I_1 - I_1 \cos\theta (v^2 - v'^2)] - \nu u J_3 \sinh p\beta, \\
B_\varphi^c &= \nu u (v^2 - v'^2)^{-1/2} \cosh p\beta (v' I_2 - vI_1) \\
&\quad + \sinh p\beta [v(J_3 \cos\theta + J_2 \sin\theta) - v' J_1 \sin\theta - J_3].
\end{aligned} \tag{74}$$

The calculation of the magnetic field of a canonical stellarator has now been reduced exactly to the evaluation of the infinite series in the six integrals I_i and J_i $(i=1,2,3)$. By appropriate truncation of the series, any desired degree of accuracy can be obtained.

5.2. Approximate magnetic surfaces

On a more detailed examination of the magnetic field represented by (62), (71), (73) and (74), it can be seen that in certain regions such as those specified by $v \simeq v'^2$, several terms of nearly equal magnitudes are required to represent each of the integrals given by (71). Consequently, a relatively large number of terms are required to represent each component of the magnetic field to any degree of accuracy. Under these circumstances, the calculation of magnetic surfaces involves rather laborious analysis, which will be postponed for presentation elsewhere. For a simpler illustration of the properties of the above series representation, attention is here restricted to the intermediate region specified by $v'^2 \gg v \gg v'$. Physically, this region is removed from the influences of both internal and external separatrices, and mathematically, the series in (71) have dominant terms and simple approximations are possible. In this case, useful approximate expressions for the magnetic field and magnetic surfaces can be found.

On retaining only the lowest-order terms ($m = 0$) in the inverse aspect ratio, it is found from (71) that

$$I_1 = \pi^2 a_l^0, \qquad I_2 = J_3 = \pi^2 a_{l-1}^0, \qquad J_1 = J_2 = I_3 = 0, \tag{75}$$

where only the lowest-order terms in v'/v are kept (see appendix 2). From (62), (73) and (74), it can be shown that $|B_\varphi| \ll |B_\eta|, |B_\theta|$, and hence from (75) and (A2.5) one finds

$$B_\eta = -bv\left(\frac{v'}{v}\right)^l \sin\Phi, \qquad B_\theta = bv\left(\frac{v'}{v}\right)^l \cos\Phi, \qquad B_\varphi \simeq 0, \tag{76}$$

where

$$b \equiv \frac{\mu_0 I_0}{4a} \cosh p\beta \frac{2l-1}{2(l-1)}. \tag{77}$$

The magnetic field in (76) can be derived approximately from the magnetic scalar potential

$$V = (ab/l)(v'/v)^l \sin\Phi, \tag{78}$$

where $\Phi = l\theta - p\varphi$, provided $\nu/v \ll 1$. The superimposed toroidal magnetic field satisfies $huB_\varphi = \text{constant}$ and hence it contributes a component

$$B_\varphi = B_0 a/hu \simeq B_0, \tag{79}$$

where the constant $B_0 \gg b$ usually. Consider the stream function defined by

$$\Psi = \tfrac{1}{2}B_0(v'/v)^2 - (bv'^2/p)(v'/v)^l \cos\Phi. \tag{80}$$

It is straightforward to verify that this is, in fact, the stream function for the total magnetic field obtained from (76) and (79). That is, $\boldsymbol{B}\cdot\nabla\Psi = 0$ and $\Psi = \text{constant}$ describes magnetic surfaces. For $l = 2$, these are elliptical contours, for $l = 3$ these are 'rounded' triangles (see figure 3) etc.

6. Summary and discussion

From a systematic derivation of the magnetic vector potential and magnetic field in two systems of toroidal coordinates, it has been shown that toroidal coordinates have the advantage over quasi-toroidal coordinates, in that the necessary integrations can be

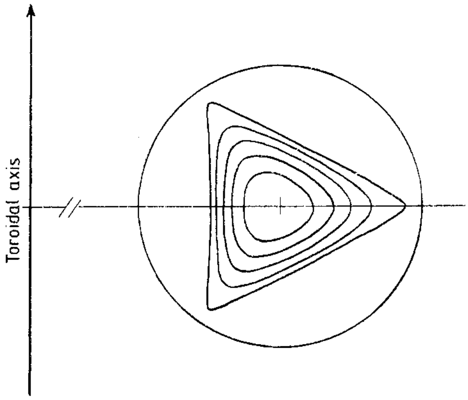

Figure 3. Magnetic surfaces specified by equation (80) for the case where $l = 3$, $p = 8$, aspect ratio $v' = 8$ and $b/B_0 = 0.05$. Only the intermediate surfaces between the inner and outer surfaces are reliable approximate magnetic surfaces for the $l = 3$ canonical stellarator field.

performed exactly by series expansions in terms of functions which are natural to the torus. The method of integration of the Biot–Savart law ensures that the boundary conditions are satisfied explicitly. This offers an alternative method to the calculation of magnetic fields due to helical currents on a torus, which has normally been done by direct numerical computations (see e.g. Gibson (1967), Gourdon *et al* (1968), Blamey *et al* (1981)). In the case of a canonical stellarator of arbitrary l, an exact analytic representation of the magnetic field has been obtained, albeit in terms of infinite series involving the generalised Legendre functions.

The discussion on helical windings showed that there is no unique definition of a uniform helix on the torus; a number of reasonable definitions are possible, depending on the system of coordinates used. Among these, the uniform helical windings defined by $d\sigma/d\varphi =$ constant (see equation (56)), being intermediate between the classical and the canonical windings, deserve some attention. For practical purposes, any regular helical winding may be regarded as a modulation of some reference uniform helical winding, such as the canonical winding.

From a simple application of the analytic representation of a canonical stellarator field, useful approximate expressions for the magnetic field and magnetic surfaces have been derived, but only in the intermediate region away from the centre and periphery of the toroidal chamber. These expressions correspond closely to those of a straight circular cylinder. By the inclusion of more terms in the series representation, further analysis can be carried out to obtain expressions which are valid for a more extended region. Further analysis and more detailed computations of various possible cases will be presented separately elsewhere.

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Appendix 1. Geometric relations

Inversion of the definitions (6) shows that

$$\cos\theta=\frac{r+R\cos\chi}{R+r\cos\chi},\qquad \sin\theta=\frac{a\sin\chi}{R+r\cos\chi},\tag{A1.1}$$

from which it follows, bearing in mind $R^2=a^2+r^2$, that

$$\mathrm{d}\theta/\mathrm{d}\chi=a/(R+r\cos\chi).\tag{A1.2}$$

Direct integration shows that

$$\theta=2\tan^{-1}\left(\frac{R-r}{a}\tan\frac{\chi}{2}\right).\tag{A1.3}$$

On the other hand, Fourier analysis of the right-hand side of (A1.2) shows

$$\frac{a}{R+r\cos\chi}=1+2\sum_{n=1}^{\infty}(-1)^n\left(\frac{R-a}{r}\right)^n\cos n\chi,\tag{A1.4}$$

from which follows, on integration,

$$\theta=\chi+2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left(\frac{R-a}{r}\right)^n\sin n\chi.\tag{A1.5}$$

Equation (57) in the text follows from the application of (A1.3) and (A1.5) to the toroidal surface $\rho=\rho_0$ or $\eta=\eta_0$, where $R=R_0$, $r=\rho_0$ and $\chi=\zeta$.

On examining triangles in figure 1, it may be shown from the sine rule that

$$(R+a)/r=[\sin(\theta+\chi-\sigma)]/\sin(\sigma-\theta)=\sin\chi\cot(\sigma-\theta)-\cos\chi\tag{A1.6}$$

and

$$r/(R-a)=(\sin\sigma)/\sin(\chi-\sigma)=\sin\chi\cot(\chi-\sigma)-\cos\chi.\tag{A1.7}$$

Since χ is arbitrary and $R^2-a^2=r^2$, one deduces $\cot(\sigma-\theta)=\cot(\chi-\sigma)$ with the appropriate solution

$$\sigma-\theta=\chi-\sigma.\tag{A1.8}$$

Equation (56) in the text follows from the application of this result to the surface of the toroidal chamber where $\chi=\zeta$.

Appendix 2. Approximate expansion coefficients

Approximate expressions for the expansion coefficients occurring in (71) can be obtained from the asymptotic properties of the generalised Legendre functions. Considering only the lowest-order terms in the inverse aspect ratio ($m=0$), then if $l\geqslant 2$, it is necessary only to obtain approximate expressions for a^0_n, with $n\neq 0$, on account of the delta functions in (71).

It can be shown (Robin 1958) from the appropriate series representations of the generalised Legendre functions that when $v\gg 1$, one has asymptotic representations

$$Q^s_{n-1/2}(v)=\frac{(-1)^s\sqrt{\pi}}{2^{n+1/2}}\frac{\Gamma(s+n+\frac{1}{2})}{\Gamma(n+1)}\frac{1}{v^{n+1/2}},\tag{A2.1}$$

and

$$P_{n-1/2}^{-s}(v)=\frac{2^{n-1/2}}{\sqrt{\pi}}\frac{\Gamma(n)}{\Gamma(s+n+\frac{1}{2})}v^{n-1/2}\qquad (n\neq 0)\tag{A2.2}$$

where s is an integer and the accuracy is $O(v^{-2})$. Application of these expressions to (45) shows

$$a_{0n}^{s}=\frac{2\sqrt{2}}{\pi}\frac{\lambda\varepsilon_s}{v'^2v^{3/2}}\left(1+\frac{1}{2n}\right)\left(\frac{v'}{v}\right)^n\qquad (n\neq 0).\tag{A2.3}$$

Now since the summation

$$\sum_{s=0}^{\infty}\varepsilon_s\exp(-s\beta)=\tfrac{1}{2}\coth\left(\frac{\beta}{2}\right),\tag{A2.4}$$

and $\beta=1/v'$ when $v\gg v'$, the sum can be approximated by v'. This result, together with the definition (68), gives

$$a_n^0=\frac{2\sqrt{2}\lambda}{\pi v'v^{3/2}}\left(1+\frac{1}{2n}\right)\left(\frac{v'}{v}\right)^n\qquad (n\neq 0).\tag{A2.5}$$

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